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Phase Transitions in Dynamical Random Graphs

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We study a large-time limit of a Markov process whose states are finite graphs. The number of the vertices is described by a supercritical branching process, and the dynamics of edges is determined by the rates of appending and deleting. We find a phase transition in our model similar to the one in the random graph model $G_{n,p}$. We derive a formula for the line of critical parameters which separates two different phases: one is where the size of the largest component is proportional to the size of the entire graph, and another one, where the size of the largest component is at most logarithmic with respect to the size of the entire graph. In the supercritical phase we find the asymptotics for the size of the largest component.

KEY WORDS: inhomogeneous random graphs; phase transitions.

1. INTRODUCTION

Graphs which are themselves undergoing local random changes in time are being intensively studied over the last few years in probability theory. This class of random processes has, besides its mathematical novelty, profound relations to computer science, physics and biology (see, e.g., ref. 8). In fact, already in the end of the 50ies Erdös and Rényi[(10)] predicted a wide range of possible applications of non-homogeneous random graphs. A general class of Markov processes on the dynamic graphs was introduced in ref. 13 along with some examples of non-homogeneous graph models (see also ref. 14). In the recent years a number of mathematical studies in this area was inspired by numeric or heuristic results from physics: these are, e.g., results on a *small-world* model (ref. 22 later ref. 3), a *scale-free* random graph (ref. 2 and later ref. 6), a *uniformly* randomly networks (ref. 7 and later refs. 9 and 5).

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We consider a Markov process with states in the space of finite graphs with multiple directed edges. The evolution of this process is described by the rates of appending new vertices and edges, and the rate of deleting edges as follows. Let $V(t)$ and $\mathcal{L}^d(t)$ denote the sets of vertices and directed edges at time $t \geq 0$, correspondingly, with $|V(0)| = 1, \quad \mathcal{L}^d(0) = \emptyset$. Here the number of the vertices $|V(t)|$, $t > 0$, is a random process itself, it is Yule process[1] with a parameter $\gamma > 0$. This means, that with every vertex in the graph we associate a Poisson process with intensity γ, every occurrence of which corresponds to the appearance of a new vertex in the graph. In particular, this implies that $\mathbf{E}|V(t)| = e^{\gamma t}$ and

$$|V(t)|\, e^{-\gamma t} \;\Rightarrow\; \xi \quad \text{as } t \to \infty,$$

where ξ follows the Exp (1)-distribution (see, e.g., ref. 1). As soon as there are at least two vertices in the graph, from each vertex we draw with intensity λ a new edge to a vertex which we choose with equal probabilities among the rest of the existing vertices in the graph. Any edge in the graph is deleted with intensity μ, i.e., the lifetime of any edge is exponentially distributed with mean value $1/\mu$. We assume that all the processes of appending and deleting are independent.

Here we study a non-directed simple (i.e., no multiple edges) graph, call it $\mathcal{G}(t)$, naturally associated with the introduced model $(V(t), \mathcal{L}^d(t))$ as follows. The set of vertices of $\mathcal{G}(t)$ is the same set $V(t)$, and there is an edge at time t between any two vertices in the graph $\mathcal{G}(t)$, if and only if there is at least one edge in the set $\mathcal{L}^d(t)$ between the same vertices. Let $\mathcal{L}(t)$ denote the set of edges of $\mathcal{G}(t)$.

This model is a certain subgraph of an original model introduced in ref. 13. It was already indicated in ref. 13 that this model behaves similar to the classical random graph $G_{n,p}$ (see ref. 4) with $p \sim c/n$. More exactly, for any fixed μ and γ, if λ is sufficiently small then the largest connected component has at most const $\times \log |V(t)|$ vertices when $t \to \infty$, while for all large values of λ the largest component has const $\times\ |V(t)|$ number of vertices as $t \to \infty$. However, the inhomogeneity of the model requires more precise analysis rather than a simple comparison with $G_{n,p}$ model. In particular, when the parameters are close to their critical values (described below), there is a positive fraction of vertices with mean value of degree greater than one as well as a positive fraction of vertices with mean value of degree less than one. Here we analyze the introduced model using mainly the analogy with certain multi-type branching processes.

The important feature of this model is that it interpolates between two known classes of models which have widely different properties. (See ref. 19 for the details.) If we do not delete the edges in our model, i.e., letting $\mu = 0$, we obtain a continuous-time generalization of the uniformly grown network introduced in ref. 7. On the other hand, when we let $\mu \to \infty$, and also $2\lambda/\mu \to c$ for some constant $c > 0$, our graph behaves in the limit as $G_{n,p}$ model with $p = c/n$. (This was already shown in ref. 19, but one can also readily see it from the result (1.6) below, which in this case recovers the known equation $\beta = 1 - e^{-c\beta}$ for the size of the largest component of $G_{n,p}$.)

A detailed study of the distribution of edges in the $\mathcal{G}(t)$ presented in ref. 18 allowed in particular to derive a formula for the mean value of the number of k-cycles in $\mathcal{G}(t)$. This led to the formula for the line of phase transition on the state space of the parameters $\gamma > 0,\ \mu > 0,\ \lambda > 0$ in ref. 19. It was conjectured there that the phase transition should be similar to the second order phase transition in $G_{n,p}$ model. Here we prove this conjecture, and find the asymptotics of the size of the largest component in the supercritical case. Recall that a component is any connected subgraph, which is not connected to any other vertex in the rest of the graph, and the size of component is the number of vertices in it.

To formulate our result we define for any $\gamma > 0$ and $\mu > 0$

$$g(t, \gamma, \mu) = \begin{cases} \frac{e^{(1-\frac{\mu}{\gamma})t} - 1}{\gamma - \mu}, & \text{if } \mu \neq \gamma, \\ t/\gamma, & \text{if } \mu = \gamma. \end{cases}$$

Then we set

$$\lambda^{cr}(\gamma, \mu) = \frac{1}{2} \sup \left\{ x > 0 : \sum_{k=2}^{\infty} x^k \mathbf{E} \prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}, \gamma, \mu) < \infty \right\} \tag{1.1}$$

where $\eta_1, \ldots, \eta_k$ are independent random variables with a common Exp (1)-distribution.

Theorem 1.1. *Let $\gamma > 0$ and $\mu > 0$ be fixed arbitrarily, and let $X\Big(\mathcal{G}(t)\Big)$ denote the size of the largest component in $\mathcal{G}(t)$.*

(I) If $\lambda < \lambda^{cr}(\gamma, \mu)$ then there exists a constant $c = c(\lambda)$ such that

$$\mathbf{P}\{X(\mathcal{G}(t)) > c \log |V(t)|\} \to 0 \quad \text{as } t \to \infty.$$

(II) If $\lambda > \lambda^{cr}(\gamma, \mu)$ then for any $\varepsilon > 0$

$$\mathbf{P}\left\{ \left| \frac{X(\mathcal{G}(t))}{|V(t)|} - \beta \right| < \varepsilon \right\} \to 1 \quad \text{as} \quad t \to \infty, \tag{1.2}$$

where

$$\beta = \int_0^{\infty} \widetilde{\beta}(s) e^{-s} ds \tag{1.3}$$

with a function $\widetilde{\beta}(s)$, $s > 0$, *defined as a positive solution to the following equation*

$$1 = \widetilde{\beta}(s) + \exp\left\{-2\lambda \int_0^\infty g(s \wedge \tau, \gamma, \mu)\widetilde{\beta}(\tau)e^{-\tau}d\tau\right\}. \quad (1.4)$$

Remark 1.1 *The constant* β *can be written in another way, namely,*

$$\beta = \int_0^\infty \beta(s)\gamma e^{-\gamma s}ds \quad (1.5)$$

with a function $\beta(s)$, $s > 0$, *defined as a positive solution to the following equation*

$$1 = \beta(s) + \exp\left\{-2\lambda \int_0^s e^{-\mu\tau} \int_\tau^\infty \beta(v)\gamma e^{-\gamma(v-\tau)}dv d\tau\right\}. \quad (1.6)$$

Remark 1.2 *The critical value* $\lambda^{cr}(\gamma, \mu)$ *is the smallest positive root of*

$$H(x) = 1 + \sum_{n=1}^\infty (-1)^n \left(\frac{2x}{\mu}\right)^n \frac{1}{n!}\left[\prod_{l=1}^n \frac{1}{1+(l-1)\mu/\gamma}\right]. \quad (1.7)$$

It is worth noticing a scaling property of our model, namely that a graph $\mathcal{G}(t) = \mathcal{G}(t, \gamma, \mu)$ has the same distribution as a graph $\mathcal{G}(t/\gamma, 1, \mu/\gamma)$.

As it was already observed in, ref. 19 this model when $\mu = 0$ behaves essentially as a uniformly grown network in ref. 7. The last one is a discrete time model. Clearly, one can reformulate our model and study a discrete time analogue for all $\mu > 0$ as well. Other related models were introduced and analyzed in refs. 15 and 16 and 17 The continuous time models seem to be more natural for the applications, e.g., in social science or biology (see ref. 21).

2. PROOF

2.1. Preliminary Results About the Graph $\mathcal{G}(t)$

Here we recall the results of ref. 18 which we need in our proof. Consider the graph $\mathcal{G}(t)$. Set $\tau_1 = 0$ and call τ_n, $n \geq 2$, the consecutive moments of jumps of the process $|V(t)|$, so that

$$|V(\tau_n)| - |V(\tau_n -)| = 1 \quad \text{and} \quad |V(\tau_n)| = n.$$

Further we shall write

$$V(t) = \left\{v_0, v_{\tau_2}, \ldots, v_{\tau_{|V(t)|}}\right\}, \quad (2.1)$$

where for each vertex v_s index s denotes the moment of appearance of this vertex in the graph.

Let us record a result from ref. 18 on the "most probable trajectory" of $(|V(s)|, 0 \leq s \leq t)$. For any $S > 0$ and $T > \Delta > 0$ define a set of trajectories

$$\mathcal{A}(S, T, \Delta) \tag{2.2}$$
$$= \left\{(|V(s)|, 0 \leq s \leq T + S) : \left| \frac{|V(S + (k+1)\Delta)|}{|V(S + k\Delta)|} - (1 + \gamma\Delta) \right| \leq \Delta^{3/2}, \right.$$
$$\left. 0 \leq k \leq \left[\frac{T}{\Delta}\right] - 1 \right\}.$$

Then as it was proved in ref. 18 for any $n \geq 1$

$$\mathbf{P}\{\mathcal{A}(S, T, \Delta) \,|\, |V(S)| = n\} \geq \left(1 - \frac{\gamma}{\Delta^2 n}\right)^{[\frac{T}{\Delta}]}. \tag{2.3}$$

This says that after have reached a large value (n), the number of vertices increases almost deterministically, namely exponentially with rate γ. Making use of the known results on Yule process (ref. 1, p. 109) we also get the formula

$$\mathbf{P}\{|V(S)| \geq n\} = \left(1 - e^{-\gamma S}\right)^{n-1}, n \geq 1, S > 0. \tag{2.4}$$

Combination of (2.3) and (2.4) gives us

$$\mathbf{P}\{\mathcal{A}(S, T, \Delta)\} \geq \left(1 - \frac{\gamma}{n\Delta^2}\right)^{T/\Delta} \left(1 - e^{-\gamma S}\right)^{n-1} \tag{2.5}$$

for all $n \geq 1$. Choosing now $T = t - \sqrt{t}$, $S = \sqrt{t}$, $n = e^{\gamma S/2}$ and $\Delta = n^{-1/6}$ we derive from (2.5) for

$$\mathcal{A}(t) = \mathcal{A}(\sqrt{t}, t - \sqrt{t}, e^{-\gamma\sqrt{t}/12})$$

that

$$\mathbf{P}\{\mathcal{A}(t)\} = 1 - o(e^{-t^{1/3}}), \quad \text{as } t \to \infty. \tag{2.6}$$

With a mild abuse of notation we shall write later on $V(t) \in \mathcal{A}(t)$, meaning

$$(|V(s)|, \; 0 \leq s \leq t) \in \mathcal{A}(t).$$

Denote by $p_t(v_s, v_\tau)$ the probability of an edge between the vertices v_s and v_τ in the graph $\mathcal{G}(t)$, given a set $V(t) = V$ and $v_s, v_\tau \in V$. Observe that in fact this probability depends on the entire set V. As noted in ref. 18 conditional on the set of vertices $V(t) = V$ distributions of the edges are independent. Thus we introduce independent Bernoulli random variables $\xi(v_s, v_\tau) \sim \text{Be}(p_t(v_s, v_\tau))$, $v_s, v_\tau \in V$, to represent the edges of the graph with a set of vertices $V(t) = V$. It is shown in ref. 18 that conditionally on $V(t) = V \in \mathcal{A}(t)$ one has

$$p_t(v_s, v_\tau) = 2\lambda\, g(\gamma(t - s \vee \tau), \gamma, \mu) \frac{1}{|V(t)|} \left(1 + \varepsilon(s \vee \tau, t)\right), \tag{2.7}$$

where $\varepsilon(t', t) \to 0$ when $t' \geq t^{1/2}$ and $t \to \infty$, and also

$$p_t(v_s, v_\tau) < B\lambda \ (g(\gamma(t - s \vee \tau), \gamma, \mu) + 1) \ \frac{1}{|V(t)|} \tag{2.8}$$

for some $B = B(\gamma, \mu) > 0$.

Note that when $t \to \infty$ most of the vertices v_s in our graph satisfy condition $s \geq \sqrt{t}$, more precisely according to (2.4) and the definition of $\mathcal{A}(t)$ we have

$$\mathbf{P}\{|V(\sqrt{t})| \geq |V|^{7/12} | V(t) \in \mathcal{A}(t)\} = o(1) \tag{2.9}$$

as $t \to \infty$. Therefore formula (2.7) is mostly used here.

2.2. Revealing a Connected Component

Note that the distribution of edges in graph $\mathcal{G}(t)$ is derived conditionally on the set $V(t)$. Given $V(t)$ we shall find a connected component in a sample of the correspondent random graph $\mathcal{G}(t)$. We use a well-known method of branching processes naturally associated with random graph (see, e.g., ref. 12 or more recent ref. 11). More precisely, we shall modify an algorithm from ref. 11 to take into account the non-homogeneity of our graph. Fix an arbitrary vertex $v^1 = v \in V(t)$ to be the root. Add to this root all the neighbours of v, i.e., the vertices connected to the vertex v by one edge in the graph $\mathcal{G}(t)$, denote them $W_1(v) = \{v_{s_1}, \ldots, v_{s_k}\}$. We call set $W_1(v)$ the first generation of v. Mark v as *saturated*. This finishes the first step of the algorithm resulting in a tree with a set of vertices $\{v, v_{s_1}, \ldots, v_{s_k}\}$ and a set of edges $\{(v, v_{s_1}), \ldots, (v, v_{s_k})\}$; call this tree $T_1(v)$.

Let $T_n(v)$ denote the tree we have constructed after the n-th step of this algorithm. We say that the distance between the root and any other vertex of this tree is k, if there are exactly k edges between them. All the vertices at distance k from the root we call the k-th generation of v. At the $(n+1)$-st step we choose a non-saturated vertex $v^{n+1} = v_s$ which has the largest index s among the closest to the root v vertices. Find all the neighbours of v_s among the vertices not used previously in the algorithm, call their set $W_{n+1}(v_s)$, and mark vertex v_s as saturated. Finally, add the set $W_{n+1}(v_s)$ to the set of vertices of $T_n(v)$, then add all the edges from v_s with ends in the set $W_{n+1}(v_s)$ to the set of edges of $T_n(v)$, and call the new graph $T_{n+1}(v)$. This finishes the $(n+1)$-st step of our algorithm. Continue this process until we end up with a tree consisting of saturated vertices only, call it $T(v)$. Observe, that given a graph $\mathcal{G}(t)$ and a vertex v the tree $T(v)$ is defined uniquely.

Clearly, the number of offspring we assign to a vertex $v^n = v_s^n$ at the n-th step depends on the set of vertices that has been used, and on the vertex v_s^n itself, more exactly, on s. Call this number $\zeta_n(v_s^n) = \zeta_n(v_s^n, \mathcal{G}(t))$.

2.3. Subcritical Case: $\lambda < \lambda^{cr}(\gamma, \mu)$

We shall prove the first part of Theorem 1.1. Here we explore a multi-type branching process approach. (Notice that our method when applied to the $G_{n,p=c/n}$ model with $c < 1$ provides a short and a simple way to get an optimal upper bound for the size of the largest component.[20])

Fix $\gamma > 0$, $\mu > 0$ and $\lambda < \lambda^{cr}(\gamma, \mu)$ arbitrarily. We shall use the following obvious bound

$$\mathbf{P}\{X(\mathcal{G}(t)) \geq N \mid V(t) = V\} \leq \sum_{v \in V} \mathbf{P}\{|T(v)| \geq N \mid V(t) = V\}, \tag{2.10}$$

provided $\mathbf{P}\{V(t) = V\} > 0$.

Let us define now a multi-type branching process where the individuals are independent of others, with V being the set of all possible types in the population. Any individual of type v produces an offspring of any type u with probability $p_t(v, u)$, independent of producing other types, but only one of each type. Suppose, we have at generation zero one ancestor of type v fixed arbitrarily. Set $Y_0(v) = 1$, and let $Y_n(v)$, $n \geq 1$, denote the number of offspring of an individual of type v in the n-th generation. Clearly,

$$\mathbf{P}\{|T(v)| \geq N \mid V(t) = V\} \leq \mathbf{P}\left\{\sum_{k=1}^{|V|} Y_k(v) \geq N - 1\right\} \tag{2.11}$$

for any $N > 0$. Let $n > 1$ be fixed arbitrarily. We can bound the right-hand side of (2.11) using the generalized Chebyshev inequality:

$$\begin{aligned}
\mathbf{P}\left\{\sum_{k=1}^{|V|} Y_k(v) \geq N - 1\right\} &\leq \mathbf{P}\left\{\sum_{i=1}^{n-1} Y_i(v) \geq \frac{N-1}{n+1}\right\} \\
&\quad + \sum_{j=0}^{n-1} \mathbf{P}\left\{\sum_{i=0}^{|V|} Y_{n+j+in}(v) \geq \frac{N-1}{n+1}\right\} \\
&\leq h^{-\frac{N-1}{n+1}} \left(\mathbf{E}h^{\sum_{i=1}^{n-1} Y_i(v)} + \sum_{j=0}^{n-1} \mathbf{E}h^{\sum_{i=0}^{|V|} Y_{n+j+in}(v)}\right)
\end{aligned} \tag{2.12}$$

for any $h \geq 1$. In the following we will show that for some finite n and $h > 1$ each of the generating functions in (2.12) is uniformly bounded in v and V.

We start with the following proposition, which together with its counterpart for the supercritical case (Proposition 2.2 below) sheds a light on the transition at λ^{cr} in the structure of graph.

Proposition 2.1. *For any $\lambda < \lambda^{cr}$ there exist finite n_0, $T > 0$ and $c < 1$ such that for all $k \geq n_0$, and for all $t \geq T$ and $V \in \mathcal{A}(t)$*

$$\mathbf{E}Y_k(v) \leq c < 1, \quad v \in V. \tag{2.13}$$

Proof: First we fix t, $V \in \mathcal{A}(t)$, $v \in V$ and $k \geq 3$ arbitrarily. Let then $\xi(u, v)$, $u \neq v, u, v \in V$, denote Bernoulli random variables $\text{Be}(p_t(u, v))$, independent for different pairs (u, v). Let further $\xi_n(u, v)$, $n \geq 1$, be independent copies of $\xi(u, v)$. We shall also set $\xi_n(v, v) \equiv 0$ for all $v \in V$, $n \geq 1$. Then for any $n \geq 1$ we have

$$Y_n(v) =_d \sum_{u^1 \in V} \xi_1(v, u^1) Y_{n-1}(u^1) =_d \sum_{u^1,\ldots,u^n \in V^n} \xi_1(v, u^1) \ldots \xi_n(u^{n-1}, u^n). \tag{2.14}$$

Hence,

$$\mathbf{E}Y_k(v) = \sum_{u^1,\ldots,u^k \in V} p_t(v, u^1) \ldots p_t(u^{k-1}, u^k). \tag{2.15}$$

Define now

$$\theta(t, \gamma, \mu) = g(\gamma t, \gamma, \mu) e^{-\gamma t}.$$

Assume, $v = v_{s_0}$ for some $0 < s_0 = t - \tau_0 < t$. Consulting for the details ref. 18 one can find with the help of (2.7) that (2.15) equals

$$\mathbf{E}Y_k(v_{s_0}) = (2\lambda\gamma)^k \int_0^t \ldots \int_0^t \left(\prod_{i=1}^{k} \exp\{-\gamma\,(s_{i-1} \vee s_i)\} \right.$$
$$\left. \times\, \theta(t - (s_{i-1} \vee s_i), \gamma, \mu) e^{\gamma s_i} \right) ds_k \ldots ds_1 + \varepsilon(t), \tag{2.16}$$

where $\varepsilon(t) \to 0$ as $t \to \infty$. For simplicity we shall write further $\theta(t, \gamma, \mu) = \theta(t)$ and $g(t, \gamma, \mu) = g(t)$. Making a change of variables $\gamma(t - s_i) \to s_i$ in the integral we rewrite the last formula as

$$\mathbf{E}Y_k(v_{t-\tau_0}) = (2\lambda)^k \int_0^{\gamma t} \ldots \int_0^{\gamma t} g(\gamma\tau_0 \wedge s_1)\, e^{-s_1} \left(\prod_{i=2}^{k} g(s_{i-1} \wedge s_i)\, e^{-s_i} \right)$$
$$\times\, ds_k \ldots ds_1 \; + \varepsilon(t)$$
$$= (2\lambda)^k\, \mathbf{E} g(\gamma\tau_0 \wedge \eta_1) \prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}) + \varepsilon(t), \tag{2.17}$$

where $\eta_1, \ldots, \eta_k$ are independent random variables with a common Exp(1)-distribution.

Let us fix now $s > 0$ arbitrarily and compare two functions $F(k)$ and $F(s,k)$ defined as follows

$$F(k) = \mathbf{E} \prod_{i=1}^{k-1} g(\eta_i \wedge \eta_{i+1}), \quad k \geq 2,$$

and

$$F(s,2) = \mathbf{E}\, g(s \wedge \eta_2), \quad F(s,k) = \mathbf{E}\, g(s \wedge \eta_2) \prod_{i=2}^{k-1} g(\eta_i \wedge \eta_{i+1}), \quad k > 2.$$

Straightforward computations yield

$$F(2) = \frac{1}{\mu + \gamma} \quad \text{and} \quad F(s,2) = \frac{1}{\mu}\left(1 - e^{-\frac{\mu}{\gamma}s}\right). \tag{2.18}$$

Note that for all $k \geq 2$ we have

$$F(s,k+1) = \int_0^\infty g(s \wedge t) e^{-t} F(t,k)\, dt. \tag{2.19}$$

We claim that

$$\begin{aligned} F(s,k) &= \sum_{n=1}^{k-2} (-1)^{n+1} \left(\frac{1}{\mu}\right)^n \frac{1}{n!} \left[\prod_{l=1}^{n} \frac{1}{1 + (l-1)\,\mu/\gamma}\right] F(s,k-n) \\ &\quad + (-1)^k \left(\frac{1}{\mu}\right)^{k-1} \frac{1}{(k-1)!} \left[\prod_{l=1}^{k-1} \frac{1}{1+(l-1)\,\mu/\gamma}\right] \left(1 - e^{-(k-1)\frac{\mu}{\gamma}s}\right) \\ &=: \sum_{n=1}^{k-2} b_n\, F(s,k-n) + b_{k-1} a_k(s) \end{aligned} \tag{2.20}$$

for all $k \geq 2$ (setting a sum over an empty set of indices to be zero). Indeed, in the case $k = 2$ this is given by (2.18). Let $m \geq 2$ and assume that (2.20) holds for all $2 \leq k \leq m$. Then applying the linear integral operator with kernel $g(s' \wedge s)e^{-s}$ to both sides of (2.20) with $k = m$ and using (2.19), we derive that (2.20) holds for $k = m + 1$ as well. This proves our claim.

Now taking into account (2.20) and relations

$$F(k+1) = \int_0^\infty \int_0^\infty e^{-t} g(t \wedge s) e^{-s} F(s,k)\, ds\, dt$$

for all $k \geq 2$, we derive also

$$F(k) = \sum_{n=1}^{k-2} b_n\, F(k-n) + b_{k-1} a_k, \tag{2.21}$$

where

$$a_k = \int_0^\infty e^{-s}\left(1 - e^{-(k-1)\frac{\mu}{\gamma}s}\right) ds.$$

Consider now

$$\widetilde{F}(x) := \sum_{k=3}^{\infty} F(k)x^k$$

and

$$\widetilde{F}(s, x) := \sum_{k=3}^{\infty} F(s, k)x^k.$$

From (2.21) we derive

$$\widetilde{F}(x)\left(1 - \sum_{k=1}^{\infty} b_k x^k\right) = F(2)x^2\left(\sum_{k=1}^{\infty} b_k x^k\right) + \left(\sum_{k=3}^{\infty} b_{k-1} a_k x^k\right) \qquad (2.22)$$

for all x such that $\widetilde{F}(x)$ converges, and similarly from (2.20) we derive

$$\widetilde{F}(s, x)\left(1 - \sum_{k=1}^{\infty} b_k x^k\right) = F(s, 2)x^2\left(\sum_{k=1}^{\infty} b_k x^k\right) + \left(\sum_{k=3}^{\infty} b_{k-1} a_k(s) x^k\right). \qquad (2.23)$$

Recall that according to the definition (1.1) of λ^{cr} the power series $\widetilde{F}(x)$ converges for all $0 < x < 2\lambda^{cr}$. This due to (2.22) (notice also that all $0 < a_k < 1$) is equivalent to the condition

$$\min\left\{x > 0 : \sum_{k=1}^{\infty} b_k x^k = 1\right\} = 2\lambda^{cr}.$$

(Hence, formula (1.7) follows.) Since also all $0 \leq a_k(s) < 1$, this together with (2.23) allows us to conclude that for all $0 < x < 2\lambda^{cr}$ the series $\widetilde{F}(s, x)$ converge uniformly in $s \geq 0$. This implies that for any $\lambda < \lambda^{cr}$ there are some positive constants A and α such that

$$(2\lambda)^k F(s, k) < Ae^{-\alpha k}$$

uniformly in $s \geq 0$. Substituting this into (2.17) we readily get the statement of the proposition. □

Now we shall study the properties of the generating functions of $Y_n(v)$, $v \in V$, $n \geq 1$, call them

$$[\mathbf{g}_{(n), V}(h)](v) = \mathbf{E}h^{Y_n(v)}, \qquad h \geq 0.$$

According to (2.14)

$$[\mathbf{g}_{(1),V}(h)](v) = \prod_{u\in V} (1 + p_t(v,u)(h-1))\,. \tag{2.24}$$

We shall also define $\mathbf{g}_{(1),V}$ as an operator on the space of positive vector-functions $f = (f(u), u \in V)$, so that

$$[\mathbf{g}_{(1),V}(f)](v) = \prod_{u\in V}(1 + p_t(v,u)(f(u)-1)), \quad v \in V.$$

Then by recurrence we have for any constant $h > 0$ and $n \geq 1$

$$\left[\mathbf{g}_{(n),V}(h)\right](v) = \left[\mathbf{g}_{(1),V}(\mathbf{g}_{(1),V}(\ldots(h)))\right](v). \tag{2.25}$$

Observe an obvious property of monotonicity of the operator $\mathbf{g}_{(n),V}$: whenever $f_1 \leq f_2$, i.e., if $f_1(u) \leq f_2(u)$ for all $u \in V$, then

$$\mathbf{g}_{(n),V}(f_1) \leq \mathbf{g}_{(n),V}(f_2). \tag{2.26}$$

Proposition 2.1 and the properties of generating functions imply

$$\frac{\partial}{\partial h}\left[\mathbf{g}_{(k),V}(h)\right](v)\Big|_{h=1} = \mathbf{E}Y_k(v) \leq c < 1 \tag{2.27}$$

for all $k \geq n_0$.

Let

$$Z_k(v) = \sum_{i=1}^{k} Y_i(v),\, k \geq 1,$$

and $Z_0(v) \equiv 0$. From now on we fix $n = n_0$ with the constant n_0 defined in Proposition 2.1. We shall use property (2.27) of operator $\mathbf{g}_{(n_0),V}$ to continue the bound (2.12), which we rewrite in new notations as

$$\mathbf{P}\left\{Z_{|V|}(v) \geq N-1\right\} \leq h^{-\frac{N-1}{n+1}}\left(\mathbf{E}h^{Z_{n-1}(v)} + \sum_{j=0}^{n-1}\mathbf{E}h^{\sum_{i=0}^{|V|} Y_{n+j+in}(v)}\right). \tag{2.28}$$

Let us fix $0 \leq j \leq n-1$ arbitrarily and consider for $k \geq 1$

$$\begin{aligned} Z_{k,j}(v) = \sum_{i=0}^{k} Y_{n+j+in}(v) =_d &\sum_{(u_1,\ldots,u_{n+j})\in V^{n+j}} \xi_1(v,u_1)\ldots\xi_{n+j}(u_{n+j-1},u_{n+j}) \\ &\times\left(1+\widetilde{Z}_k(u_{n+j})\right), \end{aligned} \tag{2.29}$$

where $\xi_i(v,u)$, $\widetilde{Z}_k(w)$, $v, u, w \in V$, $i \geq 1$, are independent, and

$$\widetilde{Z}_k(w) =_d \sum_{i=1}^{k} Y_{in}(w).$$

Set also $\widetilde{Z}_0(w) \equiv 0$.

Lemma 2.1. *There exists $h_0 > 1$ such that for all $1 \leq h \leq h_0$*

$$\mathbf{E}h^{\widetilde{Z}_k(v)} < C \tag{2.30}$$

for some constant C uniformly in $k \geq 1$, $v \in V$, and $V(t) = V \in \mathcal{A}(t)$, $t \geq T$.

Proof: Write for $k \geq 1$

$$\widetilde{Z}_k(v) =_d \sum_{(u_1,\dots,u_n)\in V^n} \xi_1(v, u_1)\dots\xi_n(u_{n-1}, u_n)(1 + \widetilde{Z}_{k-1}(u_n)),$$

where $\xi_i(v, u)$, $1 \leq i \leq n$, $\widetilde{Z}_{k-1}(w)$, $v, u, w \in V$, are independent. This gives us the following recurrent formula

$$[\widetilde{\mathbf{g}}_k(h)](v) \equiv \mathbf{E}h^{\widetilde{Z}_k(v)} = \left[\mathbf{g}_{(n),V}\left(h[\widetilde{\mathbf{g}}_{k-1}(h)]\right)\right](v). \tag{2.31}$$

Set further

$$||\mathbf{g}_{(n),V}(h)|| = \max_{v\in V}[\mathbf{g}_{(n),V}(h)](v).$$

Since for each $v \in V$ the generating function $[\mathbf{g}_{(n),V}(y)](v)$, $y \geq 0$, is convex and increasing in $y \geq 0$, $||\mathbf{g}_{(n),V}(y)||$ is also convex in $y \geq 0$. By (2.27) we have

$$\frac{\partial}{\partial h}[\mathbf{g}_{(n),V}(h)](v)\Big|_{h=1} \leq c < 1. \tag{2.32}$$

One can also show, using again (2.7), that for any fixed $h_1 > 1$ and all $h \in [1, h_1]$

$$\frac{\partial^2}{\partial h^2}[\mathbf{g}_{(n),V}(h)](v) \leq K \tag{2.33}$$

for some $K > 0$. Both (2.32) and (2.33) hold uniformly in $V(t) = V \in \mathcal{A}(t)$, $t \geq T$, and $v \in V$. All these facts, namely, convexity and the last two bounds, imply the existence of a unique $h_0 = h_0(V) > 1$, which is the largest number such that the line y/h_0 has a (unique) common point $y_0 = y_0(V)$ with the curve $||\mathbf{g}_{(n),V}(y)||$, $y > 0$. Observe that since the bounds (2.32) and (2.33) are uniform, we have $\liminf_{V\in\mathcal{A}(t),\ t\geq T} h_0(V) = \bar{h} > 1$, and also $\limsup_{V\in\mathcal{A}(t),\ t\geq T} y_0(V) = \bar{y} < \infty$ due to formula (2.7).

Clearly, $||\mathbf{g}_{(n),V}(y_0)|| \geq 1$. This yields

$$||\mathbf{g}_{(n),V}(y_0)|| = y_0/h_0 \geq 1. \tag{2.34}$$

Now due to monotonicity property (2.26) and by (2.34) we have for all $1 \leq h \leq h_0$

$$||\mathbf{g}_{(n),V}(h)|| \leq ||\mathbf{g}_{(n),V}(h_0)|| \leq ||\mathbf{g}_{(n),V}(y_0)|| = y_0/h_0, \tag{2.35}$$

and thus again by the monotonicity for all $1 \le h \le h_0$

$$||\mathbf{g}_{(n),V}(h[\mathbf{g}_{(n),V}(h)])|| \le ||\mathbf{g}_{(n),V}\left(h\,\frac{y_0}{h_0}\right)|| \le ||\mathbf{g}_{(n),V}(y_0)|| = y_0/h_0, \tag{2.36}$$

implying due to (2.31)

$$[\widetilde{\mathbf{g}}_k(h)](v) \le y_0/h_0, \tag{2.37}$$

for all $1 \le h \le h_0$. This proves (2.30), since for all $V \in \mathcal{A}(t),\ t \ge T$, we have $y_0/h_0 \le \bar{y}/\bar{h}$. □

With a help of bound (2.30) we derive further from (2.29)

$$\mathbf{E}h^{Z_{k,j}(v)} = [\mathbf{g}_{(n+j),V}(h[\widetilde{\mathbf{g}}_k(h)])](v) \le [\mathbf{g}_{(n+j),V}(h\,y_0/h_0)](v) \le C_1 \tag{2.38}$$

for all $0 \le j \le n-1$ and $1 \le h \le h_0$ uniformly in $k \ge 1$, $v \in V$, $V \in \mathcal{A}(t)$ and $t > T$, which together with (2.28) yield

$$\mathbf{P}\left\{Z_{|V|}(v) \ge N-1\right\} \le h_0^{-\frac{N-1}{n+1}}\,(C_2 + nC_1) \tag{2.39}$$

for some constant $C_2 > 0$, and all $V \in \mathcal{A}(t)$ and $t > T$. Now we use this bound in (2.11), and substituting the result into (2.10) we finally get

$$\mathbf{P}\{X(\mathcal{G}(t)) \ge N \mid V(t) = V\} \le |V|\,h_0^{-\frac{N-1}{n+1}}\,(C_2 + nC_1) \tag{2.40}$$

for all $V \in \mathcal{A}(t)$ and $t > T$. It is clear, that for some constant $c > 0$ whenever $N \ge c\log|V|$, the right-hand part of (2.40) goes to zero as $|V| \to \infty$. Now taking into account the definition of $\mathcal{A}(t)$ together with asymptotics (2.6) and (2.4), we readily get the first statement of Theorem 1.1.

2.4. Supercritical Case: $\lambda > \lambda^{cr}(\gamma, \mu)$

Now we turn to the second part of our theorem. We shall follow the branching processes approach used in ref. 11, but taking into account the non-homogeneity of our model.

2.4.1. Uniqueness of a Giant Component

First we will show that the probability of having in $\mathcal{G}(t)$ a component of a size between $k_- := a\log|V(t)|$ and $k_+ := |V(t)|^{2/3}$, where a is some positive constant, goes to zero as $t \to \infty$. To simplify further notations let $\mathcal{C}(t)$ denote a connected component in graph $\mathcal{G}(t)$, as well as its set of vertices. We want to bound

$$P(t) := \mathbf{P}\{\text{there exists } \mathcal{C}(t) \text{ with } k_- \le |\mathcal{C}(t)| \le k_+\}.$$

Clearly, due to (2.4) and (2.6) we have for any fixed $\delta > 0$

$$
\begin{aligned}
P(t) = \mathbf{P}\{&\text{there exists } \mathcal{C}(t) \text{ with } k_- \le |\mathcal{C}(t)| \le k_+ \text{ and } \mathcal{C}(t) \cap V(t-\delta) \ne \emptyset \\
&\mid V(t) \in \mathcal{A}(t) \text{ and } |V(t)| > e^{\gamma\sqrt{t}/2}\} \\
&+ \mathbf{P}\{\mathcal{C}(t) \cap V(t-\delta) = \emptyset \text{ for any } \mathcal{C}(t) \text{ with } k_- \le |\mathcal{C}(t)| \le k_+ \\
&\mid V(t) \in \mathcal{A}(t) \text{ and } |V(t)| > e^{\gamma\sqrt{t}/2}\} + o(1) \\
=:\ & P_1(t) + P_2(t) + o(1),
\end{aligned}
\tag{2.41}
$$

as $t \to \infty$.

Consider first the second term $P_2(t)$. Observe, that formula (2.7) gives us conditionally on $V(t) = V$ with $V \in \mathcal{A}(t)$ the following bound for all $t - \delta \le s \le t$ and $u \in V$

$$
p_t(v_s, u) \le \frac{C\lambda}{|V|}\,\delta, \tag{2.42}
$$

where C is some positive constant independent of choice of V. From now on fix

$$
\delta = (2C\lambda)^{-1}. \tag{2.43}
$$

Recall that in classical $G_{n,p}$ model with $p = 1/(2n)$ the largest connected component is at most $a_2 \log n$ for some absolute constant a_2, with probability tending to one as $n \to \infty$. This together with bound (2.42) implies that any connected component of $\mathcal{G}(t)$ if it consists only of the vertices of the set $V(t) \setminus V(t-\delta)$, has with a high probability (as $t \to \infty$) size at most $a_2 \log |V(t)|$. Hence, if the constant a in the definition of k_- is such that $a \ge 2a_2$, and $\delta = (2C\lambda)^{-1}$ we have

$$
P_2(t) = o(1) \tag{2.44}
$$

as $t \to \infty$.

To bound $P_1(t)$ we shall use an algorithm similar to the one introduced in Sec. 2.2. Let us fix

$$
V \in \mathcal{A}(t) \quad \text{with} \quad V(t-\delta) = V_\delta \quad \text{and} \quad |V| > e^{\gamma\sqrt{t}/2} \tag{2.45}
$$

arbitrarily. Conditionally on $V(t) = V$ we shall reveal in graph $\mathcal{G}(t)$ a connected component which contains vertex v as follows. Let us fix a number $n \ge 1$ (to be chosen later on). Then at the first step we reveal not only all the neighbours of $v = v^1$, but also all the vertices at distance up to n from v^1. As previously, we begin with the vertices which are closest to v^1 and have the largest index (i.e., the youngest in the graph). Call v saturated and proceed to the second step. At each $k > 1$ step find among the non-saturated revealed vertices a vertex v_s closest to the root v and with the largest index s, call it v^k. If none of non-saturated revealed vertices is left we stop the algorithm. Otherwise, reveal all the vertices at distance n from v^k. (Notice, that all the vertices at distance less than n from v^k are already revealed at the previous step.) Then call v^k saturated, and proceed to the $k+1$ step.

In particular, if $n = 1$, this algorithm repeats the one in Sec. 2.2.

Let $T_k^n(v)$ denote a tree after the k-th step of this new algorithm, and let $|T_k^n(v)|$ denote the total number of vertices in this tree (including the non-saturated). Notice, that according this definition $|T_k^n(v)|$ contains at least k different (saturated) vertices. Otherwise, $T_k^n(v)$ is not defined.

It is easy to see that for any $n \geq 1$ and $c_0 > 0$

$$\mathbf{P}\{ \text{ there exists } \mathcal{C}(t) \text{ with } k_- \leq |\mathcal{C}(t)| \leq k_+ \text{ and } \mathcal{C}(t) \cap V(t-\delta) \neq \emptyset \tag{2.46}$$

$$| \;\; V(t) = V \text{ and } V(t-\delta) = V_\delta\}$$

$$\leq k_+ \max_{k_- \leq k \leq k_+} \mathbf{P}\{\#\{ \text{ non-saturated vertices in } T_k^n(v)\} < c_0 k \text{ for some } v \in V(t-\delta)\}$$

$$\leq k_+ \max_{k_- \leq k \leq k_+} |V| \max_{v \in V_\delta} \mathbf{P}\{\#\{ \text{ non-saturated vertices in } T_k^n(v)\} < c_0 k\}.$$

Let $\{v = v^1, \ldots, v^k\}$ be the consecutively saturated vertices in $T_k^n(v)$, and let then $\zeta_{n,i}(v^i)$, $1 \leq i \leq k$, denote the number of offspring of the n-th generation of a saturated vertex v^i. We shall use the following result.

Proposition 2.2. *For any $\lambda > \lambda^{cr}$ there exist finite n, $T > 0$ and $c > 1$ such that for all $t \geq T$, $V \in \mathcal{A}(t)$ with $V(t-\delta) = V_\delta$, and $v \in V_\delta$*

$$\mathbf{E} \sum_{i=1}^{k} \zeta_{n,i}(v^i) \geq ck \quad \text{for any} \quad 1 \leq k \leq k_+ = |V|^{2/3}. \tag{2.47}$$

The proof of this result is very similar to the proof of Proposition 2.1, therefore we omit it here for the sake of brevity. We shall only explain why we need here a condition $v \in V(t-\delta)$. Recall that the main term of the asymptotics of (2.16) as $t \to \infty$ is

$$(2\lambda)^k F(\gamma(t-s_0), k),$$

which is zero if $s_0 = t$. To bound it by a positive constant from below, we consider only $0 \leq s_0 \leq t - \delta$.

Lemma 2.2. *Under conditions (2.45) let $T_k^n(v)$ denotes a tree with n defined in Proposition 2.2. There exist positive constants a_1 and c_0 (independent of V and*

V_δ) such that for all

$$a_1 \log |V| \leq k \leq k_+ = |V|^{2/3} \tag{2.48}$$

we have

$$|V| k_+ \max_{v \in V_\delta} \mathbf{P}\left\{ \#\left\{ \textit{non-saturated vertices in } T_k^n(v) \right\} < c_0 k \right\} \to 0, \tag{2.49}$$

as $|V| \to \infty$.

Proof: Let us fix $v \in V_\delta$ arbitrarily. Let $\zeta_{l,1}(v^1)$, $1 \leq l < n$, denote the number of offspring of the l-th generation of a saturated vertex $v = v^1$ in our process $T_k^n(v)$. Clearly,

$$|T_k^n(v)| = 1 + \sum_{l=1}^{n-1} \zeta_{l,1}(v^1) + \sum_{i=1}^{k} \zeta_{n,i}(v^i),$$

and the number of non-saturated vertices in $T_k^n(v)$ equals $|T_k^n(v)| - k$. Then we derive for any $c_0 > 0$

$$\mathbf{P}\left\{ |T_k^n(v)| - k < c_0 k \right\} = \mathbf{P}\left\{ 1 + \sum_{l=1}^{n-1} \zeta_{l,1}(v^1) + \sum_{i=1}^{k} \zeta_{n,i}(v^i) < k + c_0 k \right\}. \tag{2.50}$$

Proposition 2.2 allows us to bound the last probability as follows

$$\begin{aligned} &\mathbf{P}\left\{ |T_k^n(v)| - k < c_0 k \right\} \\ &\leq \mathbf{P}\left\{ \sum_{i=1}^{k} \zeta_{n,i}(v^i) < \mathbf{E}\left\{ \sum_{i=1}^{k} \zeta_{n,i}(v^i) \right\} - (c - 1 - c_0) k \right\}. \end{aligned} \tag{2.51}$$

Next we define for all $m \geq 1$

$$\zeta_{m,i}^{+}(v^i) = \sum_{(u_1, \ldots, u_m) \in V^m} \xi_i^1(v^i, u_1) \ldots \xi_i^m(u_{m-1}, u_m),$$

where $\xi_i^j(u, v)$ are independent copies of the random variables $\xi(u, v)$ introduced above. Notice that for all $1 \leq i \leq k$

$$\mathbf{E}\zeta_{n,i}(v^i) \leq \mathbf{E}\zeta_{n,i}^{+}(v^i) \leq C_1 \tag{2.52}$$

for some constant C_1 independent of V, which follows from asymptotics (2.17). Choose from now on $c_0 = (c - 1)/2$. Then using the condition $k < |V|^{2/3}$ together with (2.7)–(2.8) and (2.52) we derive

$$\mathbf{P}\left\{\sum_{i=1}^{k}\zeta_{n,i}(v^i) < \mathbf{E}\left\{\sum_{i=1}^{k}\zeta_{n,i}(v^i)\right\} - (c-1-c_0)k\right\}$$
$$\leq e^{-kc_1} + \mathbf{P}\left\{\sum_{i=1}^{k}\zeta^+_{n,i}(v^i) < \mathbf{E}\left\{\sum_{i=1}^{k}\zeta^+_{n,i}(v^i)\right\} - (c-1)k/4\right\} \quad (2.53)$$

for some $c_1 > 0$ and all large k. The technique of concentration inequalities (see, e.g., ref. 11 Sec. 2.1) enables one to bound the last probability from above by $e^{-c_2 k}$ for all large k and some positive constant c_2. Using this in (2.51) we derive for any $a_1 \log|V| \leq k \leq k_+$

$$|V|\, k_+ \mathbf{P}\left\{|T^n_k(v)| - k < c_0 k\right\} \leq |V| k_+\, e^{-c_2\, k_-} = |V|^{5/3}\, |V|^{-c_2 a_1}. \quad (2.54)$$

Choosing now $a_1 = 2/c_2$, we readily get the statement (2.49) of the lemma.

Lemma 2.2 together with (2.46) yields that if in (2.41) we have $a_1 \log|V(t)| \leq k_- \leq k_+$ then

$$P_1(t) = o(1) \text{ as } t \to \infty. \quad (2.55)$$

Now choosing $a = \max\{a_1, 2a_2\}$ and setting

$$k_- = a \log|V(t)|$$

we have both (2.44) and (2.55). This together with (2.41) proves that the probability $P(t)$ of having in $\mathcal{G}(t)$ a component of a size between $k_- = a\log|V(t)|$ and k_+ goes to zero as $t \to \infty$.

Now we will prove that there may exist at most one component of size $|V(t)|^{2/3}$ in $\mathcal{G}(t)$ as $t \to \infty$. We shall use the idea from, ref. 11 p.110. Given $V(t) = V \in \mathcal{A}(t)$ with $V(t-\delta) = V_\delta$, where δ satisfies (2.43), suppose that we have two trees $T^n_{k_+}(v)$ and $T^n_{k_+}(v')$ (constructed as above on the graph with vertices V) with $v \in V_\delta$ and $v' \in V_\delta$. Assume these trees do not have common vertices, and their sets of the non-saturated vertices are U and U', respectively. As we just showed in the last lemma, $|U|, |U'| \geq c_0 k_+$ with a high probability. Note that the size of any connected component in $V \setminus V_\delta$ is with a high probability at most $a_2 \log|V|$ (see the argument which led to (2.44)). Hence, to contribute into a tree of a size k_+, any component in $V \setminus V_\delta$ should be connected to at least one vertex in V_δ. Recall that bound (2.42) is also valid for a probability of an edge between any vertex in $V \setminus V_\delta$ and V. Therefore the probability that there is a vertex in V_δ which is connected to more than, say $\log|V|$ different vertices in $V \setminus V_\delta$, goes to zero as $|V| \to \infty$. This gives us the following bound

$$\mathbf{P}\left\{|U \cap V_\delta| > \frac{c_0 k_+}{2a_2(\log|V|)^2}\right\} = 1 - o(1) \quad (2.56)$$

as $|V| \to \infty$, and a similar bound holds of course for U'. If $s \vee \tau \geq \sqrt{t}$, then by (2.7) we have

$$p_t(v_s, v_\tau) \geq \frac{b}{|V|} \tag{2.57}$$

for some positive constant b and all $v_s, v_\tau \in V(t-\delta)$. By (2.9) we have

$$\mathbf{P}\left\{\#\{v_s \in U : s \leq \sqrt{t}\} \geq |V|^{7/12} \;\middle|\; V(t) \in \mathcal{A}(t)\right\} \tag{2.58}$$

$$\leq \mathbf{P}\left\{|V(\sqrt{t})| \geq |V|^{7/12} \;\middle|\; V(t) \in \mathcal{A}(t)\right\} = o(1)$$

as $t \to \infty$, and similarly for U'.

Observe that the probability that there are no edges between non-saturated vertices of the trees $T^{\delta}_{k_+}(v)$ and $T^{\delta}_{k_+}(v')$ with given sets U, U' is

$$\prod_{v_s \in U, v_\tau \in U'} (1 - p_t(v_s, v_\tau)).$$

Therefore taking into account (2.56), (2.58) and (2.57) we obtain the following upper bound for the probability that there are no edges between non-saturated vertices of $T^{\delta}_{k_+}(v)$ and $T^{\delta}_{k_+}(v')$:

$$\left(1 - \frac{b}{|V|}\right)^{|V|^{7/6}} + o(1) = o(1) \quad \text{as} \quad |V| \to \infty.$$

This together with (2.4) and (2.6) implies that the probability of having more than one component of the size $|V(t)|^{2/3}$ in $\mathcal{G}(t)$ goes to zero as $t \to \infty$.

Hence, with a probability tending to one as $t \to \infty$, every vertex of a subgraph $\mathcal{G}(t)$ either belongs to some component of size at most $k_- = a \log |V(t)|$, or it belongs to a unique giant component, call it $\mathcal{C}(t)$, of size $|\mathcal{C}(t)| = X(\mathcal{G}(t)) \geq |V(t)|^{2/3}$.

2.4.2. The Expectation of the Size of Giant Component

According to (2.6) for any $\beta > 0$ and $\varepsilon > 0$ we have

$$\mathbf{P}\left\{\left|\frac{X(\mathcal{G}(t))}{|V(t)|} - \beta\right| < \varepsilon\right\} = \sum_{V \in \mathcal{A}(t)} \mathbf{P}\left\{\left|\frac{X(\mathcal{G}(t))}{|V(t)|} - \beta\right| < \varepsilon \mid V(t) = V\right\}$$

$$\mathbf{P}\{V(t) = V\} + o(1) \text{ as } t \to \infty. \tag{2.59}$$

Therefore, till the end of the proof we fix $V \in \mathcal{A}(t)$ arbitrarily, and consider all the following events and probabilities conditionally on $V(t) = V$. Correspondingly, we shall denote $\mathbf{P}\{\cdot \mid V(t) = V\} = \mathbf{P}_V\{\cdot\}$.

Define for all $v_s \in V$

$$\chi(v_s) = \begin{cases} 1, & \text{if } v_s \notin \mathcal{C}(t), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$X(\mathcal{G}(t)) = |V| - \sum_{v_s \in V} \chi(v_s). \tag{2.60}$$

Note that

$$q(t, v_s) = \mathbf{P}_V \{\chi(v_s) = 1\}$$

differs by at most $o(1)$ (as $|V| \to \infty$) from the probability, that process $T_k^n(v_s)$, $k \geq 1$, stops before it accumulates k_- vertices. Call the last probability $q'(t, v_s)$.

Let further $q^-(t, v_s)$ denote the probability of extinction of multi-type branching process $\{Y_k(v_s),\ k \geq 1,\}$ defined in Sec. 2.3. Clearly, $q^-(t, v_s) \leq q'(t, v_s)$ simply because $T_k^n(v_s)$ may contain only once any vertex in V, unlike $Y_k(v_s)$. Also for any $U \subset V$ let us introduce (similar to the process $Y_k(v_s)$) a multi-type branching process $Y_k^U(v_s), k \geq 1$, with the set of types being the set $V \setminus U$. In this process a vertex of type v produces offspring of any type u with probability $p_t(v, u)$, independent of producing other types, but only one of each type. Let further $q_U^+(t, v_s)$ denote the probability of extinction of the process $Y_k^U(v_s)$. Clearly, the introduced probabilities are related as follows

$$q^-(t, v_s) - |o(1)| \leq q(t, v_s) \leq \max_{U \subset V : |U| = |V|^{2/3}} q_U^+(t, v_s) + |o(1)|. \tag{2.61}$$

Consider first

$$q^-(t, v_s) = \prod_{v \in V} (1 - p_t(v_s, v)) + \sum_{k=1}^{|V(t)|-1} \sum_{W \subset V : |W| = k} \left(\prod_{u \in W} p_t(v_s, u) q^-(t, u) \right) \left(\prod_{u \notin W} (1 - p_t(v_s, u)) \right). \tag{2.62}$$

Using (2.7)–(2.8) we derive for all $|W| \leq |V|^{2/3}$

$$\prod_{v \in V \setminus W} (1 - p_t(v_s, v)) = \exp\left\{ -\sum_{u \in V} p_t(v_s, u) \right\} + o(1), \tag{2.63}$$

as $|V| \to \infty$. This together with (2.62) implies

$$\begin{aligned} & q^-(t, v_s) \\ &= e^{-\sum_{u\in V} p_t(v_s,u)} \left(1+\sum_{k=1}^{|V|^{2/3}} \sum_{0<s_1<s_2<\ldots<s_k<t: v_{s_i}\in V} \prod_{i=1}^{k} p_t(v_s, v_{s_i}) q^-(t, v_{s_i})\right) + o(1) \\ &= e^{-\sum_{u\in V} p_t(v_s,u)} \left(1+\sum_{k=1}^{|V|^{2/3}} \frac{1}{k!} \left(\sum_{u\in V} p_t(v_s, u) q^-(t, u)\right)^k\right) + o(1) \\ &= e^{-\sum_{u\in V} p_t(v_s,u)(1-q^-(t,u))} + o(1) \end{aligned} \tag{2.64}$$

as $|V| \to \infty$.

From (2.64) we derive with the help of (2.7)–(2.8), (2.2) and (2.6) that

$$q^-(t, v_s) = q^-(t, s) + o(1)$$

as $t \to \infty$, where function $q^-(t, s)$ satisfies for all $0 \le s \le t$

$$\begin{aligned} q^-(t, s) &= \exp\left\{-\int_0^t \left(1-q^-(t,\tau)\right) 2\lambda\, g(t-s\vee\tau)\frac{\gamma e^{\gamma\tau}}{e^{\gamma t}} d\tau\right\} + o(1) \qquad (2.65) \\ &= \exp\left\{-2\lambda \int_0^t \left(1-q^-(t,\tau)\right) \frac{e^{(\gamma-\mu)(t-s\vee\tau)}-1}{\gamma-\mu}\, \gamma e^{-\gamma(t-\tau)} d\tau\right\} + o(1). \end{aligned}$$

Similarly, one can get for all $U \subset V$ with $|U| < |V|^{2/3}$

$$q_U^+(t, v_s) = q^-(t, s) + o(1)$$

as $|V| \to \infty$. This together with (2.61) implies in turn

$$q(t, v_s) = q^-(t, s) + o(1), \tag{2.66}$$

as $|V| \to \infty$.

Now we consider

$$\mathbf{E}_V X(\mathcal{G}(t)) = |V| - \sum_{v_s\in V} q(t, v_s) = |V| \sum_{v_s\in V} \frac{1}{|V|}(1-q(t, v_s)).$$

This with the help of (2.2), (2.6) and (2.66) leads to

$$\mathbf{E}_V X(\mathcal{G}(t)) = |V| \left(\int_{\sqrt{t}}^t \gamma e^{-\gamma(t-s)}(1-q^-(t, s))ds + o(1)\right) \tag{2.67}$$

as $t \to \infty$, $|V| \to \infty$.

Setting $\beta_t(s) = 1 - q^-(t, s)$, we get from (2.65)

$$1 = \beta_t(s) + \exp\left\{-2\lambda \int_0^t \beta_t(\tau)\frac{e^{(\gamma-\mu)(t-s\vee\tau)}-1}{\gamma-\mu}\, \gamma e^{-\gamma(t-\tau)} d\tau\right\} + o(1) \tag{2.68}$$

as $t \to \infty$. After some straightforward computation we can rewrite (2.68) as follows

$$1 = \beta_t(s) + \exp\left\{-2\lambda \int_s^t e^{-\mu(t-\tau)} \int_0^\tau \beta_t(v)\gamma e^{-\gamma(\tau-v)} dv d\tau\right\} + o(1). \quad (2.69)$$

Replacing s by $t - s$ in the last formula we get

$$1 = \beta_t(t-s) + \exp\left\{-2\lambda \int_{t-s}^t e^{-\mu(t-\tau)} \int_0^\tau \beta_t(v)\gamma e^{-\gamma(\tau-v)} dv d\tau\right\} + o(1) \quad (2.70)$$

$$= \beta_t(t-s) + \exp\left\{-2\lambda \int_0^s e^{-\mu\tau} \int_\tau^t \beta_t(t-v)\gamma e^{-\gamma(v-\tau)} dv d\tau\right\} + o(1),$$

as $t \to \infty$. From here it follows, that for all $t > s + C$ and all $C > 0$ we have

$$1 = \beta_t(t-s) + \exp \left\{-2\lambda \int_0^s e^{-\mu\tau} \left(\int_\tau^{C+\tau} \beta_t(t-v)\gamma e^{-\gamma(v-\tau)} dv + o(e^{-\gamma C/2})\right) d\tau\right\} + o(1)$$

as $t \to \infty$. Notice also, that by its definition $\beta_t(t) \equiv 0$ for all t. Hence, for any fixed s there is

$$\beta(s) := \lim_{t\to\infty} \beta_t(t-s),$$

which satisfies

$$1 = \beta(s) + \exp\left\{-2\lambda \int_0^s e^{-\mu\tau} \int_\tau^\infty \beta(v)\gamma e^{-\gamma(v-\tau)} dv d\tau\right\}. \quad (2.71)$$

Setting now

$$\widetilde{\beta}(s) = \beta(s/\gamma),$$

we derive from (2.68) and (2.71) that $\widetilde{\beta}(s)$ satisfies Eq. (1.4), which is

$$1 = \widetilde{\beta}(s) + \exp\left\{-2\lambda \int_0^\infty \widetilde{\beta}(\tau) g(\tau \wedge s, \gamma, \mu) e^{-\tau} d\tau\right\}. \quad (2.72)$$

Furthermore, it follows from the approximation by the supercritical multi-type branching processes that $\widetilde{\beta}$ must be positive.

Next we shall shortly explain that Eq. (2.72), equivalently (1.4), has a unique positive (for all $s > 0$) solution for any fixed $\lambda > \lambda^{cr}$. This will allow us to derive from (2.67) using (2.71) (or (2.72) that

$$\mathbf{E}_V X(\mathcal{G}(t)) = |V|\,(\beta + o(1)) \quad (2.73)$$

as $|V| \to \infty$, with β defined as in (1.5) (or in (1.3)).

2.4.3. Uniqueness of Positive Solution to (1.4)

Let us rewrite Eq. (2.72) as

$$f(s) = 1 - e^{-2\lambda \mathbf{A}[f](s)} =: \mathbf{H}[f](s), \quad s \geq 0, \tag{2.74}$$

where

$$\mathbf{A}[f](s) = \int_0^\infty g(s \wedge \tau, \gamma, \mu) f(\tau) e^{-\tau} d\tau.$$

First we observe that any positive function which satisfies (2.72) (or (2.74)) belongs to the following class of functions

$$\mathcal{K} := \{(h(s), s \geq 0) : h \in C^2, \ h'(s) > 0, \ h(0) = 0, \ \lim_{s\to\infty} h(s) \leq 1\},$$

where

$$\mathbf{H} : \mathcal{K} \to \mathcal{K}. \tag{2.75}$$

Introduce a norm $||h|| = sup_{s\geq 0}\, h(s)$ on $\mathcal{K}$.

Considering asymptotics similar to (2.17) and exploiting the properties of function $g(t) = g(t, \gamma, \mu)$ (in particular, that $g'(0) = 1/\gamma > 0$), one can show that for any $\lambda > \lambda^{cr}$ and for any $h \in \mathcal{K}$ there exist $C > 1$ and positive constants c and c_1 such that

$$cC^k h(s) \leq (2\lambda\, \mathbf{A})^k [h](s) \leq c_1 (2C)^k ||h|| \tag{2.76}$$

for all $k \geq 1$. The last upper bound implies in particular, that

$$\mathbf{H}[h](s) = 2\lambda\, \mathbf{A}[h](s) + O\left(||h||^2\right) \tag{2.77}$$

for all $s > 0$, when $||h|| \to 0$.

Making use of (2.77) and the lower bound from (2.76) one can show that there exists a function $h_0 \in \mathcal{K}$ such that for some finite $k > 0$

$$\mathbf{H}^k[h_0](s) \geq h_0(s), \quad s \geq 0, \tag{2.78}$$

and furthermore, for any function $h \in \mathcal{K}$ such that $h(s) \leq h_0(s), s \geq 0$, there exists also a finite n such that

$$\mathbf{H}^n[h](s) \geq h_0(s), \quad s \geq 0. \tag{2.79}$$

Observe that $\mathbf{H}$ is monotone, i.e., if $h_1(s) \geq h_2(s) \geq 0$ for all $s \geq 0$ then also

$$\mathbf{H}[h_1](s) \geq \mathbf{H}[h_2](s)$$

for all s. This together with (2.78), (2.79) and (2.75) yields existence for all $h \leq h_0$

$$\lim_{k\to\infty} \mathbf{H}^k[h](s) = \lim_{k\to\infty} \mathbf{H}^k[h_0](s) =: f_0(s) \quad \text{for all } s \geq 0. \tag{2.80}$$

Hence, by the construction f_0 is a positive solution to (2.74).

Suppose now that there exists another solution $f \in \mathcal{K}$ to (2.74), different from f_0. First we will show that in this case f should satisfy $f(s) \geq f_0(s)$ for all $s \geq 0$.

Assume that on the contrary, some $f \in \mathcal{K}$ satisfies (2.74) but $f(y) < f_0(y)$ for some positive y. Notice that we can always find a (small) function $h \in \mathcal{K}$ such that

$$h(s) \leq \min\{f(s), f_0(s), h_0(s)\}, \quad s \geq 0.$$

Then due to monotonicity of $\mathbf{H}$

$$\mathbf{H}^k[h](y) \leq \mathbf{H}^k[f](y) = f(y) \tag{2.81}$$

for all $k \geq 1$. But as $k \to \infty$ the left-hand side in (2.81) converges to $f_0(y)$ due to (2.80). Hence, our assumption $f_0(y) > f(y)$ leads to a contradiction in (2.81).

We conclude, that if there is a positive solution f to (2.74) then

$$f \in \mathcal{K}_0 := \{f \in \mathcal{K} : f \geq f_0\}.$$

Observe, that for any function $h \in \mathcal{K}$ we have for all k

$$\begin{aligned} f_0(s) - \mathbf{H}^k[h](s) &= \mathbf{H}[f_0](s) - \mathbf{H}^k[h](s) \\ &= e^{-2\lambda\, \mathbf{A}[f_0](s)}\{2\lambda\, \mathbf{A}[f_0 - \mathbf{H}^{k-1}[h]](s) + O(||f_0 - \mathbf{H}^{k-1}[h]||^2)\} \end{aligned} \tag{2.82}$$

when $||f_0 - \mathbf{H}^{k-1}[h]|| \to 0$. Hence, convergence (2.80) tells us about the properties of the (linear) operator applied to $f_0 - \mathbf{H}^{k-1}[h]$ in the principal term on the right-hand side of (2.82). We conclude that convergence (2.80) which holds for all $h \leq f_0$, should also hold whenever $h \in \{f \in \mathcal{K} : ||f - f_0|| \leq \varepsilon\}$ with some positive ε. Furthermore, consider now a similar decomposition for any $h, f \in \mathcal{K}_0$

$$\begin{aligned} &\mathbf{H}^k[h](s) - \mathbf{H}^k[f](s) \\ &\quad = e^{-2\lambda\, \mathbf{A}[f](s)}\{2\lambda\, \mathbf{A}[\mathbf{H}^{k-1}[h] - \mathbf{H}^{k-1}[f]](s) + O(||\mathbf{H}^{k-1}[h] - \mathbf{H}^{k-1}[f]||^2)\} \end{aligned}$$

when $||\mathbf{H}^{k-1}[f] - \mathbf{H}^{k-1}[h]|| \to 0$. Observe that here, since $f \geq f_0$ we have

$$e^{-2\lambda\, \mathbf{A}[f](s)} \leq e^{-2\lambda\, \mathbf{A}[f_0](s)}.$$

Therefore convergence in (2.80) also implies that there is a uniform positive ε such that if $h, f \in \mathcal{K}_0$ and $||f - h|| \leq \varepsilon$ then

$$\mathbf{H}^k[h](s) - \mathbf{H}^k[f](s) \to 0,$$

as $k \to \infty$. From here one can derive that f_0 is the only fixed point on $\mathcal{K}$ for the operator $\mathbf{H}$ since $\mathcal{K}$ consists of uniformly bounded functions.

We conclude that f_0 is a unique positive solution to (2.74).

2.4.4. The Size of the Giant Component

For any fixed $\varepsilon > 0$ we have due to the Chebyshev's inequality

$$\mathbf{P}_V\{|X(\mathcal{G}(t)) - \mathbf{E}_V X(\mathcal{G}(t))| > \varepsilon |V(t)|\} \leq \frac{\mathbf{Var}_V X(\mathcal{G}(t))}{\varepsilon^2 |V|^2}, \tag{2.83}$$

where according to (2.60) the variance

$$\mathbf{Var}_V X(\mathcal{G}(t)) = \mathbf{E}_V \Big(\sum_{v_s \in V} \chi(v_s) \Big)^2 - \Big(\mathbf{E}_V \sum_{v_s \in V} \chi(v_s) \Big)^2. \tag{2.84}$$

Consider first

$$\mathbf{E}_V \Big(\sum_{v_s \in V} \chi(v_s) \Big)^2 = \sum_{v_s \in V} \mathbf{E}_V \chi(v_s) \left(\sum_{v_\tau \in C(v_s)} \chi(v_\tau) + \sum_{v_\tau \notin C(v_s)} \chi(v_\tau) \right), \tag{2.85}$$

where $C(v)$ denotes a (random) connected component which includes the vertex v. Recall, that according to our results the size of $C(v)$ is at most k_-, unless it is the largest component in which case $\chi(v) = 0$. Hence, only the small components $C(v)$ contribute in (2.85), and we have

$$\begin{aligned} &\mathbf{E}_V \Big(\sum_{v_s \in V} \chi(v_s) \Big)^2 \\ &= \sum_{v_s \in V} \mathbf{E}_V \chi(v_s) \left(|C(v_s)| + \mathbf{E}_V \left\{ \sum_{v_\tau \in V \setminus C(v_s)} \chi(v_\tau) \mid C(v_s) \right\} \right) \\ &= \sum_{v_s \in V} \mathbf{E}_V \chi(v_s) \left(o(|V|) + \sum_{v_\tau \in V} (\mathbf{E}_V \chi(v_\tau) + o(1)) \right) \\ &= \left(\mathbf{E}_V \sum_{v_s \in V} \chi(v_s) \right)^2 + o(|V|)\, \mathbf{E}_V \sum_{v_s \in V} \chi(v_s) \end{aligned} \tag{2.86}$$

as $|V| \to \infty$. Making use of (2.86) in (2.84) we obtain

$$\begin{aligned} \mathbf{Var}_V X(\mathcal{G}(t)) &= \Big(1 + o(|V|)\Big)\, \mathbf{E}_V \sum_{v_s \in V} \chi(v_s) = \Big(1 + o(|V|)\Big)\, \mathbf{E}_V \Big(|V| - X(\mathcal{G}(t))\Big) \\ &= (1 + o(|V|))\,(1 - \beta + o(1))\,|V|, \end{aligned} \tag{2.87}$$

as $|V| \to \infty$, where the last equality is due to (2.73). Substituting (2.87) into (2.83) we get

$$\mathbf{P}_V\{|X(\mathcal{G}(t)) - \mathbf{E}_V X(\mathcal{G}(t))| > \varepsilon |V(t)|\} = o(1)$$

as $|V| \to \infty$, for any fixed $\varepsilon > 0$. In turn this implies

$$\mathbf{P}\left\{\left|\frac{X(\mathcal{G}(t))}{|V(t)|} - \beta\right| < \varepsilon \mid V(t) = V\right\} = 1 - o(1), \quad \text{as } |V| \to \infty, \tag{2.88}$$

which together with (2.59) and (2.6) proves the second statement (1.2) of Theorem 1.1. □

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